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Quasi-static approach to non-conservative problems of the elastic stability theory

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Abstract

An elastic system loaded with the following forces is in an equilibrium state. Stability of this state in the small is investigated under the assumption that if this state is disturbed and the system is set into motion, there appear some small forces of a type unknown in advance. Stability of non-conservative elastic systems is usually analysed on the basis of dynamic approach only, that is, by means of composing equations of disturbed motion and investigating these equations. Such an approach does not conform to the considered system as the unknown type of small forces does not permit the formulation of equations of motion. For this reason, the problem of stability analysis is defined in the article in a new fashion. A quasi-static approach based on energy considerations is worked out. The approach results in the same buckling criterion as is used by the static energy method. © 2001 Elsevier Science Ltd. All rights reserved.

Keywords: Stability; Non-conservative systems; Following forces; Criterion of buckling

1. Introduction

The elastic system loaded with the dead and following forces f, which do not explicitly depend on time and velocities, is in an equilibrium state. If buckling occurs, the system under consideration changes to a new equilibrium state or begins to oscillate with increasing amplitudes. The possibility of these two different buckling types allows, as is generally accepted, only the dynamic approach to the stability analysis (Bolotin, 1961; Volmir, 1967), which is the method of small vibrations or the dynamic energy method (Leipholz, 1977) are to be used. Both methods can be implemented if in the process of the disturbed motion, small forces do not arise or the type of these forces is known in advance.

The approach of this paper determines the load level at which the system can be deformed without any external energy input. This approach does not require information on the forces arising in the process of movement and can be used for the system under consideration. The new system stability characteristic is the result of the new approach. If the information mentioned above is available, this characteristic can be

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interpreted as the lower limit for critical loads of the group of systems differing in mass distribution and in small external resistance forces.

Further, the following notation is used. Generalized coordinates q_i specify the initial state 0 of a system. Coordinates $q_i + dq_i$ specify the new state s. ΔU is the arbitrary increment of magnitude U, probably, depending on the path of transition, $0 \to s$. δU, $\delta^2 U$ specify the first and second order upon dq_i terms of increment ΔU. Sign d specially points out that δU does not depend on the path of transition.

2. Quasi-static approach

Let 0 be the analysed equilibrium state specified by generalized coordinates q_i, S be a set of adjacent kinematically permitted states s, specified by coordinates $q_i + dq_i$. Generally speaking, states s are unbalanced.

Transition $0 \to s$ to a certain state $s \in S$ is considered. Let ΔU be the increment of the elastic energy U, ΔA be the work done by the external forces f. It is possible to realize arbitrary transition $0 \to s$ as quasi-equilibrium by means of an additional force r, which is determined by the trajectory. As in the process of this transition the energy conservation principle is valid, the work of force r is

$$A_r = \Delta U - \Delta A. \tag{1}$$

If force r can be considered as disturbance, then A_r is the work of this disturbance.

The problem of stability analysis in the small is defined in the following way. The system is considered unstable if some disturbance deforms this system without energy input. Conditions when this is possible are determined. Let the result of disturbance be transition $0 \to s$ to adjacent state s and r be the additional force that corresponds with the quasi-equilibrium transition $0 \to s$. Force r can also be considered as disturbance; A_r is the work of this force. Stability condition is introduced in the form

$$A_r = \Delta U - \Delta A > 0, \tag{2}$$

for every adjacent state s and every path $0 \to s$.

The criterion of buckling is

$$\Delta U - \Delta A = 0, \tag{3}$$

for some state s and some path $0 \to s$. ΔA depends on the path $0 \to s$.

The static energy method used for conservative systems also examines the difference $\Delta U - \Delta A$ between the elastic energy increment ΔU and the work ΔA, performed by the external forces f in the case of various transitions $0 \to s$. Buckling criterion is also given by Eq. (3). This criterion follows from Lagrange–Dirichlet's theorem, which is not valid for systems under consideration. For conservative systems, work ΔA is independent of the path of transition.

For non-conservative systems, the sufficiency of analysing only the shortest path $0 \to s$ is introduced as a postulate. It is the only postulate proposed in the present approach. Along the shortest path coordinates, q_i vary proportionally. Further, this postulate is justified in different ways, but at first, one preliminary consideration based on the properties of small vibrations of the elastic system is proposed. Differential equation of small vibrations has the solution

$$\Delta q_i(q,t) = \varphi_i(q)\exp(\lambda t). \tag{4}$$

Here, q is a set of generalised coordinates and t is the time. In the stable state, λ is imaginary and $\varphi_i(q)$ are real. So increments Δq_i estimated according to Eq. (4) vary with time in a similar manner and the path $0 \to s$ is the shortest one. (It is necessary to note that after buckling, the proportionality of Δq_i vanishes.)

If the path of transition $0 \to s$ is given, there are two possible cases that require separate discussion. Let disturbance be applied quasi-statically. Then, force r is the disturbance itself. It is natural to postulate the conservative character of such a disturbance. Then, the left-hand side of Eq. (1) does not depend upon the path of transition $0 \to s$, and there is no necessity to especially postulate the proportionality of $\mathrm{d}q_\mathrm{i}$.

Let disturbance be applied dynamically. Then, force r also takes into account inertial forces. The postulation of accelerations constancy along the transition $0 \to s$ into adjacent state s results in the constancy of the inertial forces. Then, their work does not depend on the path of transition again.

According to the virtual work principle, first order terms of $\mathrm{d}q_\mathrm{i}$ in Eq. (3) collectively are equal to zero, i.e., Eq. (3) results in the energy criterion of buckling in its ordinary form,

$$\mathrm{d}^2U - \delta^2 A = 0, \tag{5}$$

for some $\mathrm{d}q_\mathrm{i}$, compatible with the geometrical boundary conditions.

Another way of constructing criterion (5) is based on the fundamental thermodynamic principle of maximal work. This principle considers a system which in its forced or spontaneous unbalanced evolution passes through a sequence of states s and produces work $-A_a$ (work performed upon the system is assumed to be positive). According to this principle, if the system is taken through this sequence in the quasi-static manner for any element of process work $-A_\mathrm{q}$ produced by the system would be greater than $-A_a$. So for any spontaneous transition $0 \to s$,

$$A_\mathrm{q} - A_a < 0. \tag{6}$$

If spontaneous transition to any $s \in S$ is impossible, the initial state is stable. The condition of stability assumes the form

$$A_\mathrm{q} - \Delta A - A_r \geqslant 0, \quad s \in S. \tag{7}$$

Here, work A_r of power disturbance r, which acts in the course of the transition, is singled out as a special term. ΔA is the actual work of load f. The formulation of the stability problem in the small predetermines negligibly the small magnitude of work A_r; so the term A_r can be omitted.

According to the first law of thermodynamics, for the elastic system work A_q is equal to the elastic energy change ΔU. Thus, Eq. (7) can be rewritten as

$$\Delta U - \Delta A > 0, \quad s \in S. \tag{8}$$

As the term A_r is omitted, inequality (7) becomes strict. Work ΔA is calculated ignoring the work of disturbance, but still depends on the path of transition. (One can point out that Eq. (8) evidently extends to elastic–plastic systems.)

The principle of maximal work from all possible paths of transition $0 \to s$ singles out the shortest one in which increments $\mathrm{d}q_\mathrm{i}$ change proportionally to each other. Indeed, let the pair of states 0, s be considered and the path of transition $0 \to s$ be represented by the sequence of steps $0 \to 1 \to 2 \to \cdots \to n \to s$ $(0, 1, 2, n \in S)$ with the proportional change of $\mathrm{d}q_\mathrm{i}$ along each step. In accordance with the verbal formulation of the principle in spontaneous transition, condition (7) is kept at each step. Consequently, impossibility of transition $0 \to 1$ means impossibility of transition $0 \to s$. The necessity for analysing steps $1 \to 2, \ldots, n \to s$ no longer arises. The possibility of $0 \to 1$ transition means instability of the system, but this fact will be determined as the pair of states 0, 1 is analysed. Thus, postulating the shortest path of transition $0 \to s$ may be considered as the consequence of the maximal work principle.

As stability is analysed in the small, only ratios of $\mathrm{d}q_\mathrm{i}$ are important. So further, in Eq. (8), the set S of adjacent states s is replaced by the set of available directions s.

According to the virtual work principle, the left-hand side of Eq. (8) does not include components of $\mathrm{d}q_\mathrm{i}$ order. If in Eq. (8) only the components of the second order are kept, left-hand sides of conditions (8) and

(5) become identical. The difference between Eqs. (8) and (5) is the explicit representation in Eq. (8) of set S. Only after this representation, condition (8) transforms into the stability criterion.

Particular motion equations take into consideration one or another peculiarity of perturbed motion and restrict thereby set S. This is seen, for instance, from the examples given below. If limitations of set S are taken into account, criterion (8), generally, leads to the same result as the dynamic one. If the limitations are not taken into account, it is natural to interpret the critical load given by Eq. (8) as the lower limit of critical loads for the group of systems, differing in mass distribution and in small resistance forces, arising in the process of movement. Another interpretation of the result is the critical load for a system whose mass is such that the inertial forces, arising in the process of movement, are small compared with disturbing forces.

Publications devoted to non-conservative problems of elastic stability discuss effects caused by small viscosity of material (internal friction). Condition (8) considers a material as ideally elastic and requires modification to take viscosity of a material into account. So, at present, only external friction (arbitrary small forces arising in the process of movement) is taken into consideration.

2.1. Quasi-static criterion of buckling

If set S coincides with set S_0 of all kinematically permitted directions, condition (8) gives the lower limit for critical loads and assumes the form,

$$\mathrm{d}^2U - \delta^2A = 0 \quad \text{for some } s \in S_0. \tag{9}$$

Here, the left-hand side of the equation is the quadratic form of independent variables $\mathrm{d}q_\mathrm{i}$, and the equation itself is identical to condition (5).

Because criterion (9) takes into consideration small forces of the type unknown in advance as well as every kind of inertial forces, critical force, derived on the basis of Eq. (9), does not depend on mass distribution.

The result of criterion (9) can be considered also as the lower limit of critical loads for the group of systems, differing in mass distribution.

Criterion (9) gives the value of the critical force only. One is to use the dynamic approach to derive the form of buckling and the disturbance that causes buckling.

2.2. Criterion for the known disturbed motion type

Let constraint of set S_0 imposed by kinematic equations be taken into account. Then, criterion (8) assumes the form

$$\mathrm{d}^2U - \delta^2A = 0 \quad \text{for some } s \in S_\mathrm{d}. \tag{10}$$

Here $S_\mathrm{d} \subset S_0$ is a subset of directions compatible with admissible motion trajectories. Subset S_d not only depends on the load level, but it also depends on the mass distribution.

As is shown in this paper with the help of examples, in elementary cases to construct set $S_\mathrm{d} \subset S_0$, it is sufficient to know only the form of kinematic equations. In the general case, the calculation of natural frequencies is required to construct S_d, and criterion (10) itself transforms into a more bulky variant of dynamic criterion.

Further, the examples demonstrate main features of the new approach. These examples consider a straight rod of length l with constant rigidity EI and use dimensionless longitudinal coordinate $\bar{x} = x/l$.

Analysed equilibrium state 0 is given by generalized coordinates $q_\mathrm{i} = 0$, so increments $\mathrm{d}q_\mathrm{i}$ are equal to values of q_i in the perturbed state. In state 0, elastic energy $U_0 = 0$, so its increment $\Delta U = U$. Therefore, further, signs δ, d in stability criterion are omitted.

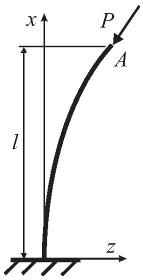

Fig. 1. The rod loaded with tangent force P.

3. Examples

(1) The rod is loaded on its upper end A with tangent force P (Fig. 1). Flexure is permitted only on the figure plane.

Quasi-static critical load: Relative deflection w (divided by the rod's length l) sets the shape of the deflection curve,

$$w(\bar{x}) = w(\bar{x}) = f_1(\bar{x}) + \sum_{i=2}^{n} b_i f_i(\bar{x}). \tag{11}$$

Here n is the number of degrees of freedom. Functions $f_i(x)$ satisfy kinematic boundary conditions

$$f_i(0) = f_i'(0) = 0. \tag{12}$$

Power conditions

$$f_i''(1) = f_i'''(1) = 0 \tag{13}$$

are not taken into account as small disturbances applied to the upper end are permitted. Coefficients b_i ($i = 2, \ldots, n$) determine possible directions of transitions. Any values of b_i are possible, so set S_0 members are various combinations of b_i. The quantities are

$$U = \frac{EI}{2l}\int_0^1 [w''(\bar{x})]^2 \mathrm{d}\bar{x} = \frac{EI}{2l}u, \tag{14}$$

$$A = \frac{Pl}{2}\left\{\int_0^1 [w'(\bar{x})]^2 \mathrm{d}\bar{x} - w(1)w'(1)\right\} = \frac{Pal}{2}. \tag{15}$$

The second term in Eq. (15) takes account of the following component of force P.

Buckling criterion (9) results in

$$P^* = \frac{KEI}{l^2}, \quad K = \min\frac{u}{a}. \tag{16}$$

Quantities u, a depend on b_i. Designation min denotes minimizing with regard to b_i. Coefficient K defines the value of critical force.

Let, for instance, the deflection be

$$w(\bar{x}) = (b_5\bar{x}^5 + b_4\bar{x}^4 + b_3\bar{x}^3 + \bar{x}^2). \tag{17}$$

Minimizing results in

$$K = 9.875 \approx \pi^2 = 9.870 \quad (b_3 = 0.111;\ b_4 = -1.123;\ b_5 = 0.433). \tag{18}$$

In the table, some dynamically found values of K are presented for comparison. Solutions of the corresponding problems are taken from (Bolotin, 1961; Panovko and Gubanova, 1967). Case 1 – point mass is at A. Case 2 – mass is distributed uniformly. Case 3 – the rod has two point masses m_1 and m_2, located at $\tilde{x}_1 = 1$ and $\tilde{x}_2 = c$. K depends on m_1/m_2 and c. The lower limit $K = \pi^2$ is reached if $m_1/m_2 \ll 1$ and $c \ll 1$. Case 4 – mass $m = 0$ is at A, its moment of inertia $I_m \neq 0$ (two small masses a long distance away from each other). Case 5 – point mass is at A, Voigt's rheological model is used.

Case	Ideally elastic rod				Vanishing viscosity
	1	2	3	4	5
K	20.19	20.03	$>\pi^2$	π^2	10.94

Result (18) actually limits from below dynamically found values of K. There exist mass distributions that realize value (18).

Critical load for a particular system: Mass m is at point A. Its moment of inertia $I_m \neq 0$ is set by parameter $\rho = \sqrt{I_m/(Ml^2)}$. The differential equation of the deflection is (deflection and longitudinal coordinates are dimensionless)

$$w^{\mathrm{IV}} + k^2 w'' = 0, \tag{19}$$

$$k^2 = Pl^2/EI. \tag{20}$$

Kinematic boundary conditions are

$$w_{\bar{x}=0} = w'_{\bar{x}=0} = 0. \tag{21}$$

Kinematic equations give power boundary conditions:

$$EIw''_{\bar{x}=1} = -I_m l \ddot{w}'_{\bar{x}=1}, \quad EIw'''_{\bar{x}=1} = ml^3 \ddot{w}_{\bar{x}=1}. \tag{22}$$

The solution of Eq. (19) complying with condition (21) assumes the form

$$w(\bar{x}) = C(\sin k\bar{x} - k\bar{x}) + D(1 - \cos k\bar{x}), \tag{23}$$

where constants C and D are time functions.

Ratio $T = D/C$ defines the shape of the curve $w(\bar{x})$. Set S_0 includes any values of T. Set S_d includes only values that satisfy power conditions (22).

Instead of buckling criterion (10), it is more convenient to use stability criterion which, considering Eqs. (14), (15) and (23), assumes the form

$$\sin kT^2 - k\sin kT + \sin k - k\cos k > 0. \tag{24}$$

The extreme value of the left-hand side of Eq. (24), taken according to the values of T, is

$$h = (1 - k^2/4)\sin k - k\cos k. \tag{25}$$

$h(k) > 0$ in the interval $k < 3\pi/2$ that is important for the stability analysis. If $k < \pi$, $\sin k > 0$ and $h(k)$ is minimum. So, for any T, criterion (24) is satisfied and buckling is impossible. Buckling is possible for $k > \pi$. Value $k = \pi$ ($K = \pi^2$) determines the lower limit of critical loads as restriction of T values owing to power conditions (22) is not taken into consideration. This value coincides with result (18), but now this value is associated only with the systems having point masses on their upper ends, because the solution of a uniform equation (19) is used.

According to Eq. (10), stable behaviour of the system for $k > \pi$ is the result of range restriction of T values, imposed by relations (22). According to Eq. (22), allowed values of T satisfy the equality,

$$\frac{T\cos k - \sin k}{T\sin k + \cos k} = \rho^2 k^2 \frac{T\sin k - (1 - \cos k)}{T(1 - \cos k) + \sin k - k}. \tag{26}$$

Case $C = 0$ is considered as the limit for $T \to \infty$.

When $\rho = 0$, set $S_{\rm d}$ consists of two load-dependent elements:

$$S_{\rm d} = \{\mathrm{tg}\,k,\ (k - \sin k)/(1 - \cos k)\}. \tag{27}$$

When $\rho \to \infty$,

$$S_{\rm d} = \{-\mathrm{ctg}\,k,\ (1 - \cos k)/\sin k\}. \tag{28}$$

For both sets (27) and (28), the critical load corresponds with their first elements.

For $T = \mathrm{tg}\,k$, condition (24) assumes the form

$$\frac{\sin k - k\cos k}{\cos^2 k} > 0 \tag{29}$$

and results in $K = 20.19$ (case 1 of the table).

If $T = -\mathrm{ctg}\,k$, condition (24) assumes the form,

$$1/\sin k > 0 \tag{30}$$

and does not hold for $k \geqslant \pi$. The critical value $K = \pi^2$ also coincides with the dynamically found one (case 4 of the table).

With other values of ρ, the quadratic in T in Eq. (26) should be solved. The determinant of this equation

$$B(k) = [(k\rho)^2 \sin k]^2 + (k\cos k - \sin k)^2 + 2(k\rho)^2[k(2 - \cos k)\sin k - (1 - \cos k)(3 - \cos k)] \tag{31}$$

up to the factor k^2 coincides with the determinant of the frequency equation (Panovko and Gubanova, 1967). If $B < 0$, the roots of Eq. (26) and the frequency of vibration are simultaneously complex. The absence of real roots should correspond to unrestricted set S_0. Thus, together with inequality $k \geqslant \pi$, condition

$$B(k) = 0 \tag{32}$$

determines buckling. Direct check shows that if $B(k) > 0$, inequality (24) is kept, so stability holds fine. Hence, disregarding additional limitation $k \geqslant \pi$, criterion (10) leads to a more complicated form of the dynamic buckling criterion.

Influence of small resistance forces: Criterion (9) permits the rise of small forces that are unknown in advance and are caused by the process of disturbed motion. Generally speaking, the example illustrates the strong influence of such forces.

The stability of the rod shown in Fig. 2 is examined by means of the method of small vibrations. Force $q = \alpha \dot{w}_{x=l}, \alpha > 0$ of external viscous resistance is applied to the upper end of the rod. Mass m is focused at point A. Mass and viscosity are specified by parameters $\bar{m} = ml^3/EI$ and $\bar{\alpha} = \alpha l^2/EI$.

Solutions of the differential equation (19) are considered:

$$w(t, \bar{x}) = w_1(\bar{x})\exp(\lambda t). \tag{33}$$

Here λ is the complex constant and t is the time.

Deflection $w_1(\bar{x})$, which satisfies geometrical boundary conditions and conjugation conditions at $\bar{x} = a(w_{a-\varepsilon} = w_{a+\varepsilon}, w'_{a-\varepsilon} = w'_{a+\varepsilon}, w''_{a-\varepsilon} = w''_{a+\varepsilon})$, has the form

$$w_1(\bar{x}) = A(1 - \cos k\bar{x}) + B(\sin k\bar{x} - k\bar{x}) + D[\sin(k\bar{x} - ka) - (k\bar{x} - ka)]_{\bar{x}>a}. \tag{34}$$

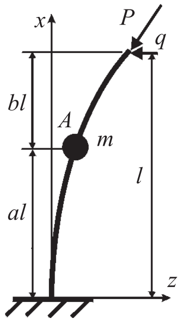

Fig. 2. Influence of small resistance force.

The boundary conditions determine complex constants A, B and C.

Shearing force Q satisfies the boundary condition

$$(Q)_{\bar{x}=a+\varepsilon} - (Q)_{\bar{x}=a-\varepsilon} = m(\ddot{w}l)_{\bar{x}=a} = -EI[(w'''/l^2)_{\bar{x}=a+\varepsilon} - (w'''/l^2)_{\bar{x}=a-\varepsilon}] \tag{35}$$

that is the constant

$$D = \frac{\tilde{m}\lambda^2}{k^3}[A(1-\cos ka) + B(\sin ka - ka)]. \tag{36}$$

The boundary conditions $(w'')_{\bar{x}=1} = 0$, $EI(w'''/l^2)_{\bar{x}=1} = \alpha(\dot{w}l)_{\bar{x}=1}$ lead to

$$A\cos k - B\sin k - D\sin kb = 0, \tag{37}$$

$$k^3(A\sin k + B\cos k + D\cos kb) = -\bar{\alpha}\lambda(A - Bk - Dkb). \tag{38}$$

The right-hand side of Eq. (38) takes into account relation (37).

If D is eliminated from Eqs. (37) and (38), the frequency determinant becomes

$$\left\| \begin{matrix} \cos k - \sin(kb)h_1\hat{m}\lambda^2 & -\sin k - \sin(kb)h_2\hat{m}\lambda^2 \\ \sin k + (\cos kb - kb\hat{\alpha}\lambda)h_1\hat{m}\lambda^2 + \hat{\alpha}\lambda & \cos k + (\cos kb - kb\hat{\alpha}\lambda)h_2\hat{m}\lambda^2 - k\hat{\alpha}\lambda \end{matrix} \right\| = 0. \tag{39}$$

Here $h_1 = 1 - \cos ka$, $h_2 = \sin ka - ka$, $\hat{m} = \bar{m}/k^3$ and $\hat{\alpha} = \bar{\alpha}/k^3$. Condition (39) results in the following cubic equation (coefficients at λ^4, λ^5 are equal to zero)

$$\hat{\alpha}\hat{m}c_3\lambda^3 + \hat{m}c_2\lambda^2 + \hat{\alpha}c_1\lambda + 1 = 0. \tag{40}$$

The coefficients are

$$c_1 = g(k), \quad c_2 = g(ka), \quad c_3 = h_1(k\sin kb - kb\sin k) + h_2(\sin(kb) - kb\cos k), \tag{41}$$

$$g(k) = \sin(k) - k\cos(k). \tag{42}$$

The system in Fig. 2 is stable if all roots of Eq. (40) have negative real parts, which are such at

$$c_1 > 0, \quad c_2 > 0, \quad c_3 > 0, \quad c_1c_2 - c_3 > 0. \tag{43}$$

According to Eq. (43), the stability of the state does not depend on absolute values of m and α. Only their presence is important.

If $\alpha = 0$ (viscous resistance is absent), then $c_1 = c_3 = 0$ and the stability condition assumes the form

$$c_2 > 0. \tag{44}$$

Critical value of force parameter k in such an ideal situation is

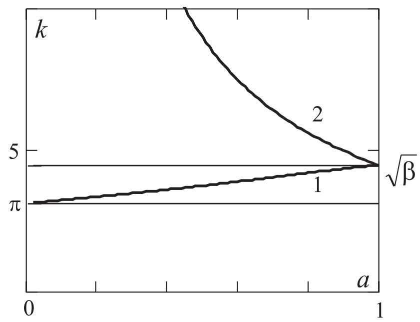

Fig. 3. *k*–*a* plot for the system on Fig. 2.

$$k_{\mathrm{id}} = \sqrt{\beta}/a. \tag{45}$$

Here $\sqrt{\beta} = 4.493$ is the root of equation $g(k) = 0$.

If $\alpha \neq 0$, the critical value of k is given by the last inequality in Eq. (43). Results are shown in Fig. 3, where digits 1, 2 mark curves for considered and ideal situations. According to this figure, the existence of external viscous forces radically changes results of stability analysis. The straight line $k = \sqrt{\beta}$ shows the case for a massless rod ($m = 0$). Energy value of the critical load (straight line $k = \pi$) restricts the dynamically found results from below.

It is possible to consider external small viscous forces as disturbances. In accordance with the example, it is necessary to take such disturbances into consideration. For this reason, it is necessary to extend the class of disturbances, used for stability analysis, by means of including in this class every possible small force, depending on system displacements (Kagan-Rosenzweig, 1999b).

The unknown type of disturbing forces does not allow one to compose equations of disturbed motion and to use the dynamic approach. The quasi-static approach is to be used for this purpose. But only the dynamic approach supplemented by the assumption on a physical nature of disturbing forces allows one to obtain the disturbance that causes loss of stability.

(2) Force P on the upper end of the rod falls behind the tangent (Fig. 4). Again, the flexure is permitted only on the figure plane.

Quasi-static critical load: The load is defined by relation (16) in which now

$$a = \int_0^1 [w'(\bar{x})]^2 \mathrm{d}\bar{x} - (1 - \xi) w(1) w'(1). \tag{46}$$

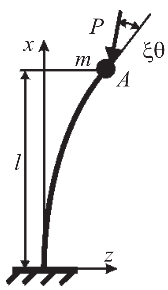

Fig. 4. Force P falls behind the tangent.

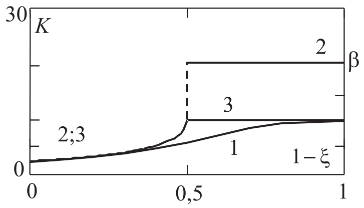

Fig. 5. K–ξ plot for the system in Fig. 4.

The result of minimizing is plotted in Fig. 5, curve 1. Curves 2 and 3 show, respectively, the results for the rod with point mass on its upper end (Dzanelidze problem) and with mass $m = 0$; $I_m \neq 0$, obtained dynamically. β denotes 20.19.

In the whole range of ξ, curves 2 and 3 differ from curve 1. Such an effect is unexpected. Actually, if $\xi \leqslant 0.5$, Euler static method and the method of small vibrations give identical results. In this range of ξ, curves 2 and 3 correspond to the static buckling mode, and it is natural to expect that criterion (9) will give the same result. The problem was considered in Kagan-Rosenzweig, 1999a) where the method of small vibrations was used and mass distribution, which responds critical curve 1, was obtained. This mass distribution conforms to a massless rod with a vertical rigid cantilever attached to its upper end and with point mass located at a relevant position on this cantilever. Thus, curve 1 does not contradict the above-mentioned interpretation of criterion (9). It is necessary to note that, according to the obtained result, stability in the Euler sense does not guarantee stability in the dynamic sense.

Critical load: For the rod, in Fig. 4, with mass m on its upper end, the moment of inertia $I_m \neq 0$. The deflection curve satisfying the differential equation of flexure has the form (23). Power boundary conditions are

$$EIw''_{\bar{x}=1}/l = -I_m \ddot{w}'_{\bar{x}=1}, \qquad EIw'''_{\bar{x}=1}/l^3 = m\ddot{w}_{\bar{x}=1} - k^2(1-\xi)w'_{\bar{x}=1}/l. \tag{47}$$

According to Eqs. (23) and (47), assumed values of T satisfy the equation

$$\frac{T\cos k - \sin k}{T\xi\sin k + \xi\cos k + 1 - \xi} = \rho^2 k^2 \frac{T\sin k - (1-\cos k)}{T(1-\cos k) + \sin k - k}. \tag{48}$$

When $\rho = 0$, the set

$$S_{\rm d} = \{\operatorname{tg} k,\ (k - \sin k)/(1-\cos k)\}. \tag{49}$$

The stability criterion is given by

$$\begin{aligned} &T(T-k)\sin k + \sin k - k\cos k + (1-\xi)\{(\sin k - k)(1-\cos k) - T[(T\sin k + 2\cos k)(1-\cos k) \\ &\quad - k\sin k]\} > 0. \end{aligned} \tag{50}$$

Substitution of $T = \operatorname{tg} k$ into Eq. (50) leads to the relation

$$\frac{\sin k - k\cos k}{\cos^2 k}[1 - (1-\xi)(1-\cos k)] > 0. \tag{51}$$

This implies the equation for critical value of force parameter k:

$$\begin{aligned} &\cos k = -\xi/(1-\xi), \quad \xi \leqslant 0.5, \\ &\sin k - k\cos k = 0, \quad \xi > 0.5. \end{aligned} \tag{52}$$

Result (52) is identical to the dynamically found result. The plot of $K(\xi)$ is shown in Fig. 5, curve 2. Curve 3 shows the result for the rod with mass $m = 0$, $I_m \neq 0$ at A.

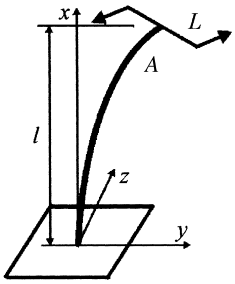

Fig. 6. The rod twisted by the tangent moment.

(3) The rod with point mass m on its upper end is twisted by the following tangential moment L. The compressive force $P = 0$ (Fig. 6). Quantities $EI_y = EI_z = EI$, EI_p represent flexural and torsional rigidities. The critical value of L is evaluated.

The rod has two degrees of freedom, and its deformation caused by perturbation can be specified either by means of small horizontal displacements u_1, v_1 of point A or by means of small horizontal forces P_1, P_2, applied to A. Values of P_1/P_2 define set S_d of directions. As kinematic equations are not used, the limit value of the critical load for systems with point masses on their upper ends is calculated.

Let $w(\bar{x}) = u + iv$ be the complex deflection and $\gamma = Ll/EI$ be the load parameter. The complex form of the differential equation for the deflection curve is (Bolotin, 1961)

$$w''' - iw'' = -p, \quad p = p_1 + ip_2 = (P_1 + iP_2)l^3/EI. \tag{53}$$

Its solution, complying with the boundary conditions $w(0) = w'(0) = w'''(1) = 0$,

$$w(\bar{x}) = (p/\gamma^2)[e^{i\gamma(\bar{x}-1)} + 0.5\gamma^2\bar{x}^2 - e^{i\gamma}(1 - i\gamma\bar{x})] \tag{54}$$

contains two constants p_1 and p_2. Any value of p_1/p_2 is considered possible.

The torque in the section $\bar{x}$ is

$$L_x(\bar{x}) = L\cos(\theta(\bar{x}) - \theta(1)) \approx L[1 - 0.5(\theta(\bar{x}) - \theta(1))^2], \quad \theta = \sqrt{u'^2 + v'^2}/l. \tag{55}$$

The twist change is

$$\chi = -\frac{Ll}{GI_p}\int_0^1 \frac{[\theta(\bar{x}) - \theta(1)]^2}{2}\mathrm{d}\bar{x}. \tag{56}$$

The work produced by the torque while deviating from the stable state,

$$A = L\chi + Lu'(1)v'(1)/l^2, \tag{57}$$

takes into account the twist change (first term) and the inclination of the plane of torque L (second term).

Elastic energy change consists of the flexural and torsional terms:

$$U = U_k + U_w, \tag{58}$$

$$U_k = L\chi, \qquad U_w = \frac{EI}{2l^3}\left[\int_0^1 (u'')^2\mathrm{d}\bar{x} + \int_0^1 (v'')^2\mathrm{d}\bar{x}\right]. \tag{59}$$

The buckling criterion (10) gives

$$0.5\left[\int_0^1 (u'')^2\mathrm{d}\bar{x} + \int_0^1 (v'')^2\mathrm{d}\bar{x}\right] - \frac{Ll}{EI} = 0 \tag{60}$$

and results in

$$(1 - \sin\gamma/\gamma)[\cos\gamma p_1^2 + (2 + \cos\gamma/\gamma)p_2^2] - (\gamma - 2\sin\gamma/\gamma)p_1 p_2 = 0 \tag{61}$$

for some p_1 and p_2.

Hence, the critical value of the load parameter γ is the root of the equation

$$(1 - \sin\gamma/\gamma)^2 \cos\gamma(2 + \cos\gamma) - (\sin\gamma - \gamma/2)^2 = 0. \tag{62}$$

This root $\gamma = 0$, i.e., instability is possible with any L.

As is known (Bolotin, 1961), the dynamic solution of the problem leads to the same result.

4. Conclusion

The article deals with stability analysis in the small of elastic non-conservative systems and contains a new definition for the problem of stability analysis: the system is considered stable if it requires an external energy input to deform it. Such a definition of the problem serves as the basis for the quasi-static approach. The approach leads to the same buckling criterion as is used by the static energy method.

The critical load value P^* found with the help of quasi-static approach takes into account small forces of non-inertial nature, which may arise in the process of movement. For this reason, P^* does not depend on mass distribution.

The load P^* can be treated also as

(a) the critical load for the system, which if disturbed undergoes small forces unknown in advance;

(b) the lower limit of dynamically found critical forces affecting the systems with different mass distributions.

Interpretation (b) is the result of the method used to base the stability criterion. It should be noted that there exists a system, found by the author (Kagan-Rosenzweig, 1998), the critical load of which, obtained by the method of small vibrations, is less than the quasi-static value P^*. This result is yet to be analysed.

The new approach gives only the value of the critical load. One is to use the dynamic approach, supplemented by the assumption on the nature of the above-mentioned small forces to derive the buckling mode and a certain disturbance that causes buckling.

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